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# Array Processing for a Multiple-Rank Signal

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This paper considers array processing for a sensor array when the signal is composed of a superposition of wave fronts. The array processing requires a multiple-signal parameter formulation. Existing array processing theory is generalized. The required signal processing functions should not be referred to as beam forming, because a matrix estimation operator is required. The rank of the estimation operator is greater than 1.

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## INTRODUCTION

Consider a large noisy room in which sound can reverberate. In one end of the room a speaker radiates a narrow band signal. In a remote part of the room a sensor array records the sound pressure at each sensor location and attempts to detect the signal. Let $k$ denote the number of sensors. Locally, the signal can cross the array in a small number of modes of travel. Let $m$ denote the number of such modes. We shall assume that $m< k$. Each mode may be associated with a separate acoustic path. In the simplest case, each mode may be represented by the time delays associated with a plane wave arrival. However, this assumption is not necessary for this discussion. All that will be assumed is that each mode is characterized by a rank-1 covariance matrix and that the vector factor of each of these rank-1 matrices is known. Let $v_i$ denote the vector factor for the $i$ th mode.

Note that since the signal is assumed to be narrow band, it is convenient to formulate the problem in the frequency domain. This means that all of the pressures discussed will be represented by complex numbers.

The signal will now be characterized by the amplitude coefficients associated with the modes. The signal field at any time is the sum of the products of each mode and its associated amplitude. Let $\alpha_i$ denote the amplitude of the $i$ th component. Then the total signal field can be described by a vector, $s$, whose components are the total signal pressures at each of the sensor positions.

$$s = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m \tag{1}$$

Let V denote a matrix whose column vectors are the vector factors.

$$V = [\ v_1 \ \ v_2 \ \ \cdots \ \ v_m\ ] \tag{2}$$

and

$$a = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_m \end{bmatrix} \tag{3}$$

so that

$$s = V\ a \tag{4}$$

So the number of rows in $V$ is $k$, the number of elements in the array, and the number of columns in $V$ is $m$. $a$ is an $m$ element complex vector.

It will also be convenient to define a covariance matrix for the complex coefficients, $\alpha_i$. Let < > denote an ensemble average. Then let

$$A \equiv \langle a a^H \rangle \tag{5}$$

Under certain assumptions, the signal covariance matrix will still be of rank-1. For example, we might require that the source not move, that none of the reflecting surfaces move, that no air currents disturb the sound path, etc. Such assumptions are too stringent for the purposes of this discussion. Instead, we shall assume that the sound paths vary enough, or at least involve enough unknown parameters, that we must attempt to estimate the $\alpha$'s, or $A$. We shall assume that the rank of $A$ is $m$.

The novelty in this paper will be the rank-$m$ signal. The conceptual problems are considerably more difficult than for the rank-1 case. Different criteria lead to different detector designs, and the connection between the designs is not obvious. The first viewpoint which we use will ignore the detection problem and concentrate on the estimation problem.

## MAXIMUM LIKELIHOOD ESTIMATION OF $a$

Maximum likelihood estimation theory seems to provide the simplest path to extend existing array processing theory for this multiple-parameter signal problem. We shall assume Gaussian noise. Let $x$ denote a column vector of complex pressures from each sensor. Let $C$ denote the noise-only covariance matrix. Then for any received pressure field, $x$, the problem is to choose $a$ to minimize

$$(x - V a)^H C^{-1} (x - V a) \tag{6}$$

This can be expanded as

$$x^H C^{-1} x - x^H C^{-1} V a - a^H V^H C^{-1} x + a^H V^H C^{-1} V a \tag{7}$$

We can rearrange the terms to get

$$x^H C^{-1} x - x^H C^{-1} V (V^H C^{-1} V)^{-1} V^H C^{-1} x \tag{8}$$

$$+ (a - (V^H C^{-1} V)^{-1} V^H C^{-1} x)^H (V^H C^{-1} V) (a - (V^H C^{-1} V)^{-1} V^H C^{-1} x)$$

Obviously, the minimum of this quantity occurs when

$$a_o = (V^H C^{-1} V)^{-1} V^H C^{-1} x \tag{9}$$

and the minimum is

$$x^H C^{-1} x - x^H C^{-1} V (V^H C^{-1} V)^{-1} V^H C^{-1} x \tag{10}$$

It is appropriate to define an estimation matrix operator as

$$E \equiv (V^H C^{-1} V)^{-1} V^H C^{-1} \tag{11}$$

This operator warrants further examination. The reader may note that $V(V^H C^{-1} V)^{-1} V^H C^{-1}$ is the projection operator for $V$ when the metric matrix is $C^{-1}$. This association with projection operators provides a nice geometric interpretation of the processing algorithms.

Note that $E$ should be used with caution when any of the $v_i$ become nearly equal. In this case, the matrix $V^H C^{-1} V$ becomes nearly singular and some undesirable noise amplification effects can occur. This subject is still under investigation.

Of course, the proper extension of conventional beam-forming can be found by setting $C$ to the identity matrix.

It is probably best not to refer to the processing scheme described above as beam-forming. All current beam-forming theory assumes that the estimation operator is a simple column vector. The fact that $E$ is a matrix complicates the processing somewhat.

## ESTIMATING THE ESTIMATOR

Assuming that the estimator matrix, $E$, is to be used, it must first be computed from $C$ and $V$. But if $C$ is to be estimated from data, it may be difficult to obtain noise-only data from which to compute $C$. Let us assume that, unbeknown to the operator, a signal is present to contaminate the estimation of $C$. Let us examine the error which will result from his attempt to evaluate Eq. 11. He will actually compute

$$E_{S+N} = (V^H (C + V\,A\,V^H)^{-1} V)^{-1} V^H (C + V\,A\,V^H)^{-1} \tag{12}$$

We must examine what happens when $C + V\,A\,V^H$ is substituted for $C$. Note that from Woodbury's formula

$$(C + V\,A\,V^H)^{-1} = C^{-1} - C^{-1} V\,T\,V^H C^{-1} \tag{13}$$

where

$$T^{-1} = A^{-1} + V^H C^{-1} V \tag{14}$$

Note that, among other things, this means that

$$(V^H (C + V\,A\,V^H)^{-1} V)^{-1} = (V^H C^{-1} V - V^H C^{-1} V\,T\,V^H C^{-1} V)^{-1} \tag{15}$$

$$= (V^H C^{-1} V)^{-1} - U$$

where

$$T^{-1} + U^{-1} = (V^H C^{-1} V) \tag{16}$$

so

$$U = - A \tag{17}$$

and

$$(V^H (C + V\ A\ V^H)^{-1} V)^{-1} = (\ V^H C^{-1} V\ )^{-1} + A \tag{18}$$

We can use this to compute the effect of the presence of signal.

$$E_{S+N} = (\ (\ V^H C^{-1} V\ )^{-1} + A)\ V^H (\ C^{-1} - C^{-1} V\ T\ V^H C^{-1}) \tag{19}$$

$$= (\ V^H C^{-1} V\ )^{-1} V^H C^{-1} - T\ V^H C^{-1} + A\ V^H C^{-1} - A\ V^H C^{-1} V\ T\ V^H C^{-1}$$

$$= (\ V^H C^{-1} V\ )^{-1} V^H C^{-1} - (\ T - A + A\ (V^H C^{-1} V)\ T\ )\ V^H C^{-1}$$

$$= (\ V^H C^{-1} V\ )^{-1} V^H C^{-1} - (\ T - A + A\ (\ T^{-1} - A^{-1})\ T\ )\ V^H C^{-1}$$

$$= (\ V^H C^{-1} V\ )^{-1} V^H C^{-1} = E$$

In other words, no error results. The formula is invariant with respect to the arrival of the signal.

## POWER ESTIMATION

The estimate of the signal pressure in the sensors is

$$s_{est} = V\ E\ x \tag{20}$$

From this, we can estimate the signal-like energy as

$$<\ (V\ E\ x)^H (V\ E\ x)> \ = \ <\ x^H E^H V^H V\ E\ x> \tag{21}$$

$$= <tr(E^H V^H V\ E\ x\ x^H)> \ = \ = \ <tr(E^H V^H V\ E\ C\ )> \ = tr(\ V^H V\ E\ C\ E^H)$$

Note that, because of the result in the previous section, we can be ambiguous about whether $E$ was estimated from the true noise- only covariance matrix or from the current data matrix. So

$$<\ s_{est}^H s_{est}> \ = tr(\ (\ V^H V\ )\ (V^H C^{-1} V)^{-1}) \tag{22}$$

This expression is evidently the proper extension of the popular adaptive bearing response algorithm, sometimes referred to as the "maximum likelihood" or "minimum variance distortionless look" algorithm.

Of course, we can also easily extend the conventional beam-forming algorithm. In this case

$$E_{\text{conv}} = (V^H V)^{-1} V^H \tag{23}$$

So

$$tr( (V^H V) E_{\text{conv}} C E^H_{\text{conv}} ) = tr( (V^H C V) (V^H V)^{-1}) \tag{24}$$

This is not, however, the optimum detection structure. There are some other conceptual issues to be discussed below.

## LIKELIHOOD RATIO DETECTION

The structure for the likelihood ratio detector is easy to derive. We need simply to look at the exponent in the ratio of the Gaussian probability functions.

$$-x^H ( C + V A V^H )^{-1} x + x^H C^{-1} x = x^H C^{-1} V T V^H C^{-1} x \tag{25}$$

If several samples of $x$ are to be averaged, this may be more conveniently written as

$$tr( C^{-1} V T V^H C^{-1} < x x^H > ) \tag{26}$$

Note that in this case, one must evidently have noise-only information from which to estimate $C$.

In fact, the form of the likelihood ratio detector suggests theoretical problems which are considerably more difficult than one might guess. The probe variable, Eq. 26, does not behave according to the simple gamma distribution which we expect for square-law detectors. It is formed from a sum of exponential variables with different variances. This, as will be seen below, complicates the conceptual picture.

## PERFORMANCE INDICES

In the previous section, we saw that the optimum detector takes the form of a quadratic inner product. For discussion of detector performance it is convenient to generalize this algorithm. We shall define a variable

$$\eta = < x^H W x > = tr( W < x x^H > ) \tag{27}$$

where $W$ is a general square matrix. This function can be regarded as a generalization of the beam-forming process. It will result in a scalar variable, $\eta$. We must now define a signal-to-noise-ratio function to describe the statistics of $\eta$. The first step is to examine the moments of $\eta$.

$$< \eta >_N = tr( C W ) \tag{28}$$

$$< \eta >_{S+N} = tr(\, (C + V\, A\, V^H)\, W\, ) \tag{29}$$

Here we encounter a subtle conceptual trap. It is tempting to use these two formulas to define a signal-to-noise ratio. This approach depends on the standard deviation of the variable being proportional to the mean value of the variable. However, since $\eta$ is a sum of unequal exponential variables, this will not give a good indicator of the standard deviation of $\eta$. We must compute the variance more carefully.

$$variance = < (\, x^H W\, x\, )^2 >_N - < \eta >_N^2 \tag{30}$$

This requires us to evaluate fourth order moments for complex Gaussian variables. It is well known that if $\xi_1$, $\xi_2$, $\xi_3$, and $\xi_4$ are real Gaussian random variables, then $< \xi_1\xi_2\xi_3\xi_4 > = < \xi_1\xi_2 > < \xi_3\xi_4 > + < \xi_1\xi_3 > < \xi_2\xi_4 > + < \xi_1\xi_4 > < \xi_2\xi_3 >$. However, the corresponding result for complex Gaussian variables is not widely known. It is

$$< \xi_1^*\xi_2\xi_3^*\xi_4 > = < \xi_1^*\xi_2 > < \xi_3^*\xi_4 > + < \xi_1^*\xi_4 > < \xi_3^*\xi_2 > \tag{31}$$

With this knowledge and some index arithmetic it can be seen that

$$variance = tr(\, W^H C\, W\, C\, ) \tag{32}$$

We are now in a position to define a signal-to-noise ratio (snr).

$$(snr)^2 = \frac{tr(\, V\, A\, V^H W\, )\; tr(\, W^H V\, A\, V^H\, )}{tr(\, W^H C\, W\, C\, )} \tag{33}$$

This is easy to compute for the likelihood ratio detector.

$$(snr)_{LR}^2 = \frac{(tr(C^{-1}V\, T\, V^H C^{-1}V\, A\, V^H\, ))^2}{tr(C^{-1}V\, T\, V^H C^{-1}C\, C^{-1}V\, T\, V^H C^{-1}C\, )} \tag{34}$$

$$(snr)_{LR}^2 = \frac{(tr(\, (V^H C^{-1}V\, )\, T\, (V^H C^{-1}V\, )\, A\, ))^2}{tr((V^H C^{-1}V\, )\, T\, (V^H C^{-1}V\, )\, T\, )} \tag{35}$$

## MAXIMIZING SIGNAL-TO-NOISE RATIO

Curiously, there is reason to believe that even Eq. 33 does not tell the whole story. To see this, consider how $W$ might be chosen to maximize the signal-to-noise ratio. We can use differential calculus, or the Schwarz inequality to maximize the signal-to-noise ratio with respect to $W$.

$$W_{SNR} = C^{-1}V\, A\, V^H C^{-1} \tag{36}$$

This produces a squared signal-to-noise ratio of

$$(snr)^2_{max} = tr(\,(V^H C^{-1} V)\,A\,(V^H C^{-1} V)\,A\,) \tag{37}$$

It is not entirely clear why this detector would not be as good as the likelihood ratio detector. It must be due to the peculiar behavior of the tail end of the distribution. This illustrates the difficulties of dealing with sums of unequal exponential variables.

Note that Eq. 37, like Eq. 22 or Eq. 24, could be used as a probe function. However, the use of fourth-order moments is not generally desirable.

## ESTIMATION OF $A$

Minimizing the right side of Eq. 37 with respect to $A$ can yield an estimate of $A$. Again, the idea is that the noise which looks most like the signal will be that which is most difficult to process against. Of course, the trace of $A$ must be constrained, so we have merely to minimize

$$\frac{tr(\,(V^H C^{-1} V)\,A\,(V^H C^{-1} V)\,A\,)}{(tr(A))^2} \tag{38}$$

We can minimize this with respect to $A$. The result is

$$A = (V^H C^{-1} V)^{-2} \tag{39}$$

However, it is not clear that this is a good estimator. Once again, the appearance of fourth-order moments may cause problems.

## ANOTHER FACTORIZATION

In all of the above discussion, the signal covariance matrix has been expressed as a product of three factors. That is, the signal covariance matrix has been written as $V A V^H$. This factorization was chosen because it lends itself to physical interpretation. However, from a mathematical viewpoint, the factorization is not unique. For some purposes, we can find a simpler factorization. If we are willing to give up the obvious physical interpretation, we can choose $V$ so that $A$ is an identity matrix and $(V^H C^{-1} V)$ is diagonal. In other words, we can assume without loss of generality that

$$A = I \quad \text{and} \quad V^H C^{-1} V = D \tag{40}$$

where $I$ is the identity matrix and $D$ is a diagonal matrix. These substitutions in the above equations provide an apparent simplification. These simplifications may help provide an intuitive understanding for readers who are comfortable with the geometric implications of Eq. 40.

## CONCLUSIONS

This paper extends the class of signals we can address with established array signal detection/estimation ideas. It does not provide a logical alternative to familiar array processing techniques. Rather, it allows extension of these signal processing ideas to a class of signal which is often present but often ignored. It provides a way to deal with problems in which multiple signals characterize a single hypothesis.

Probably the most important idea developed above is the estimator matrix, $F$, as described in Eq. 11. The utility of this estimator depends on Eq. 19. That is, we can estimate $E$ without knowing whether a signal was present in the data sampled.

The utility of probe or estimation functions such as Eq. 22, 24, 37, and 39 is not yet clear. The answer will probably depend on details of the actual applications. It is important to carefully investigate the *a priori* information about the signal parameter matrix, $A$, and the required search space.

Equation 33 can probably give a reasonable estimate of detection performance in most cases. However, since the noise is a sum of unequal variance exponential variables, it may not always give accurate results. Equation 33 should be useful if applied with caution and insight. Any analysis which ignores Eq. 33 will probably be wrong.

In general, if the problem involves a single signal which fits the usual beam-forming assumptions, the ideas presented above are not needed. If the signal breaks up into components which may not be entirely coherent, the ideas developed above may come into play.

## Appendix

## ESTIMATION MATRIX AND CONSTRAINED OPTIMAL BEAM-FORMING

The estimation matrix, $E$, can play a role in beam-forming. It occurs in the structure of optimal beam-forming with hard constraints. Since the derivation seems not to be available in the open literature, we shall examine it here.

The basic problem may be stated as follows. Design a beam-former which has minimum noise output subject to the constraint of specified responses in certain directions. Let $z$ denote a vector of beam-forming weights. Let $V$ denote a matrix whose columns are the models for signals from the constraint directions. Let $b$ denote a column vector of constraint values for each direction. The the constraint can be written as

$$V^H z = b \tag{A.1}$$

This type of constraint may be used in several ways. If the constraint directions are close together, it could be used for main-lobe maintenance or broadening. The beam could be made to have main-lobes in two or more different directions. The beam could also be made to have nulls in certain directions by setting the appropriate components of $b$ to zero. For example, if $v_1$ was the desired look direction, main-lobe maintenance could be accomplished by setting the other $v$'s to directions about $v_1$ and computing $b$ from $V^H v_1 = b$.

We shall assume that the number of constraint directions is less than the number of sensors.

The optimization process follows standard paths. First, we define a quantity to be minimized. Let $C$ denote the noise covariance matrix.

$$\mu = z^H C\, z + \lambda(V^H z - b)^H (V^H z - b) \tag{A.2}$$

We shall define $z$ as the vector which minimizes $\mu$ and find the limit as $\lambda$ goes to infinity. To simplify the notation, it is convenient to define the matrix

$$M = C + \lambda V\, V^H \tag{A.3}$$

Then we can expand the expression for $\mu$.

$$\mu = z^H (C + \lambda V\, V^H) z - \lambda z^H VB - \lambda b^H V^H z + \lambda b^H b \tag{A.4}$$

$$= (M\, z - \lambda V\, b)^H M^{-1} (M\, z - \lambda V\, b) + \lambda b^H (I - \lambda V^H M^{-1} V)\, b$$

Since the first term is the only one which depends on $z$, it is sufficient to minimize it. This means that

$$z = \lambda(C + \lambda V V^H)^{-1} V b = (\frac{1}{\lambda} C + V V^H)^{-1} V b \tag{A.5}$$

Since $V V^H$ is singular, more manipulation is required before it is clear how to take the limit as $\lambda$ goes to infinity. Again, Woodbury's theorem helps.

$$z = (\lambda C^{-1} - \lambda^2 C^{-1} V (I + \lambda V^H C^{-1} V)^{-1} V^H C^{-1}) V b \tag{A.6}$$

$$= C^{-1} V (\frac{1}{\lambda} I + V^H C^{-1} V)^{-1} b$$

Now we can take the limit as $\lambda$ goes to infinity.

$$z = C^{-1} V (V^H C^{-1} V)^{-1} b = E^H b \tag{A.7}$$

In other words, the optimal solution consists of first applying the estimation operator, $E$, and then simply multiplying by the constraint values in $b$.